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EQUIVARIANT COMPLETIONS OF RINGS WITH REDUCTIVE GROUP ACTION

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Let the connected reductive algebraic group G act on the affine variety X, over an algebraically closed field of characteristic zero. The largest G-rational submodule of the completion of the coordinate ring of X along the ideal of a closed orbit is an equivariant completion. If the orbit is a local complete intersection in X, and if H denotes its stabilizer, then it is shown that the equivariant completion is G-isomorphic to the equivariant completion of the induction from H to G of the symmetric algebra of a finite-dimensional H-module.

Introduction

All regular varieties are formally the same, in the sense that the completions of their local rings at points are always formal power series. It is possible to regard this result as part of the Cohen structure theory of complete local rings, e.g. [8, Corollary 2, p. 206]. The purpose of this work is to develop a similar sort of theory for equivariant completions of regular affine varieties with reductive algebraic group action. (Of course the completions here are at minimal stable closed subsets, i.e. closed orbits, not points.) Since we take equivariant completions, the group acts on the completion and the ring of invariants is the completion of a local ring, the ring of invariants, and thus we obtain a type of structure theorem for these, also.

Precisely, let k be an algebraically closed field of characteristic zero and X a non-singular affine k-variety on which the connected linearly reductive affine algebraic k-group G acts. Let Y be a closed orbit of G in X, let J be the ideal of Y in $k[X]$, let $B=k[Y]$ and let M be the B and G module J/J^2. Then the equivariant completion of $k[X]$ at J is shown to be the maximal rational G-submodule $^*S$ in $\prod_{n\geq 0} S^n_B(M)$. Of course Y is isomorphic to G/H, where H is the stabilizer of a point in Y. This implies that M is of the form $V|^G_H$ for some finite-dimensional H-module V and hence the ring of G invariants of $^*S$ is $\prod_{n\geq 0} S^n_k(V)^H$. On the other hand, we show that the ring of invariants is also the completion of $A=k[X]^G$ at the maximal ideal $\mathfrak{m}$ lying under J. Thus the

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completion of A at $\mathfrak{m}$ is the ring of invariants of H acting on the formal power series in a basis of V. (And H is also reductive since the homogeneous space Y is affine.)

These results are related to the normal linearization theorem of Bass and Haboush [2, Theorem 9.1], which shows $k[X]$ to be a symmetric algebra under certain assumptions, the strongest being that there is an equivariant retraction of X onto a closed subscheme containing all closed orbits. A key step in the present work is to produce a retraction onto the single closed orbit Y, not from all of X, but from its J-infinitesimal neighborhoods in X. This step is accomplished via a theory of square zero equivariant algebra extensions of algebras which, like B, have no non-trivial equivariant ideals. The main result is that all such extensions split, and this is done in Section 1 below. In Section 2 we define equivariant completions and obtain their properties, the most important being exactness. In Section 3 these results are applied to obtain the structure theorems noted above. Section 4, which is independent of the others, shows that algebras without equivariant ideals (such as those from the earlier sections) are essentially coordinate rings of affine homogeneous spaces.

The results here are also related to a technique used by Hochster and Roberts [5, Section 11, pp. 154–158]: there the associated graded ring of our equivariant completion is exploited, as well as its ring of invariants. Indeed, our notation in the proof of the local structure of invariants is chosen to mimic that of [5, (4), p. 155]. Of course the completion does have some additional functorial properties that the associated graded ring lacks.

We use strongly throughout that $k[Y]$ has no G-stable ideals. The theory of such rings is studied in [4, 7, 9]. We recall the key definitions and establish our notational conventions for use below: k and G are as above and B is a commutative k-algebra. If G acts rationally on B as algebra automorphisms, we say B is a *G-algebra*. If in addition B has no G-stable ideals except 0 and B we say B is *G-simple*. A B-module M on which G acts rationally compatibly with its action on B is called a *$B.G$-module*. For the case $B = k[G/H]$ with G/H affine the categories of $B.G$-modules and rational H-modules are equivalent [9, Theorem 3.1, p. 42; 7, Theorem, p. 656] and the functor from the latter to the former is denoted $(\ )|_H^G$, which also has a direct definition [3, p. 2]. When we need to refer to non-rational G-modules we usually call them abstract G-modules. An abstract G-module which is also a compatible B-module is called a simultaneous B and G module.

1. Equivariant algebra extensions

The purpose of this section is to establish commutative algebra extension theory where the rings have rational G-actions and the homomorphisms are G-linear. The basic theory is essentially that of extensions without group actions, as developed, for example in [8, Chapter 10], and our discussion here will be primarily a review of [8] with comments on where the G-action enters. From this basic theory, we then obtain our principal result: an extension of a G-simple algebra must be split. We begin with the basic definition.

Definition 1.1. Let B be a G-algebra. A *square zero G extension* (briefly, extension) of B is a G-algebra A and a surjective G-algebra homomorphism $\varepsilon : A \to B$ such that $(\mathrm{Ker}\ \varepsilon)^2 = 0$. The extension is trivial if there is a G-algebra homomorphism $\sigma : B \to A$ with $\varepsilon\sigma = \mathrm{id}$.

If $N = \mathrm{Ker}\ \varepsilon$ in the extension in Definition 1.1, we have the exact sequence

$$0 \longrightarrow N \longrightarrow A \longrightarrow B \longrightarrow 0 \tag{1}$$

of G-modules, and N is naturally a $B.G$-module. As a sequence of G-modules, (1) is split. Choosing a G-section s *of* ε and identifying A with $B \oplus N$ via $a \mapsto (\varepsilon a, a - s\varepsilon a)$, we find the product in $B \oplus N$ given by $(b, x)(c, y) = (bc, by + xc + f(b, c))$ where $f : B \times B \to N$ is $f(b, c) = s(b)s(c) - s(bc)$. It is easy to see that f is symmetric, k-bilinear, G-linear and satisfies the cocycle condition [8, p.179]. Conversely, if M is a $B.G$-module and $g : B \times B \to M$ is such a map, then $B \oplus M$ is a G-algebra using the above multiplication (g replacing f) and projection on B defines a square zero G-extension.

If the extension $A = B \oplus N$ is trivial, the section map σ is given by a map $t : B \to N$ with $\sigma(b) = (b, t(b))$. This map is related to f by the coboundary formula $f(a, b) = t(ab) - at(b) - bt(a)$. It is easy to check that t is G-linear, and that any such map related to f by the coboundary formula makes the extension trivial.

Ordinary extension theory for k-algebras is understood via Hochschild cohomology [8, p. 203][6, p. 283]; in the preceding paragraph we have observed that to obtain G-extensions only requires that cocycles and coboundaries must be G-linear. Formally: Let B be a G-algebra and N a $B.G$-module. Then $Z^2(B/k, N)^s$ is the set of k-bilinear symmetric maps $B \times B \to N$ satisfying the cocycle condition, $B^2(B/k, N)^s$ the image in Z^2 of the k-linear maps $B \to N$ under the coboundary formula, and $H^2(B/k, N)^s$ is the quotient Z^2/B^2. Both Z^2 and B^2 are G-modules whose G-invariants are the G-linear cocycles and coboundaries (this assertion uses that G is reductive and that B^2 is the image of $\mathrm{Hom}_k(B, N)$ under the G-linear coboundary homomorphism). A G-extension gives rise to an element of $(Z^2)^G$ and is trivial exactly when the element is in $(B^2)^G$. Thus to detect non-trivial extensions, we study the subgroup $(Z^2)^G/(B^2)^G = (H^2(B/k, N)^s)^G$ of H^2. (The fact that H^{2G} is a subgroup of H^2 also shows, as is implicit in the above discussion, that a square zero G-extension of B is trivial if and only if it is trivial as an ordinary extension.)

Associated to the extension (1) is a B-module homomorphism

$$\delta : N \to \Omega_{A/k} \otimes_A B \tag{2}$$

defined by $\delta(n) = (n \otimes 1 + 1 \otimes n) \otimes 1$, and (1) is trivial iff (2) has a left inverse [8, Theorem 58, p. 187]. We note that (2) is actually G-linear, and hence a $B.G$-module homomorphism.

We can now prove the main result of this section.

Theorem 1.2. *Let B be a G-simple G-algebra. Then every square zero extension of B is trivial.*

Proof. We will use the theorem of Doriaswamy [4, Proposition 3.5, p. 794] which shows that all $B.G$-modules are $B.G$- (and B-) projective, and hence also $B.G$-injective as well being a direct sum of $B.G$-simple modules, finitely generated over B. The square zero extensions (1) with kernel N are trivial if $(H^2(B/k, N)^s)^G = 0$. Since this cohomology group is an additive functor of N we are led to the case where N is $B.G$-simple, $B.G$-injective, and finitely generated over B. In this case (2) will have a left inverse, and hence (1) will be trivial, if δ is not zero. To see this latter, we choose a maximal ideal $\mathfrak{m}$ of A containing the annihilator of N. Localizing (1) at $\mathfrak{m}$ gives the sequence

$$0 \longrightarrow N_{\mathfrak{m}} \longrightarrow A_{\mathfrak{m}} \longrightarrow B_{\mathfrak{m}} \longrightarrow 0 \tag{3}$$

which is a square zero (ordinary) extension of $B_{\mathfrak{m}}$, and $B_{\mathfrak{m}}$ is the k-algebra obtained from B by localizing at its maximal ideal $\mathfrak{m}/N$. Since the non-singular locus of B is open, non-empty, and G-stable, and B is G-simple, $B_{\mathfrak{m}}$ is regular. Thus it is formally smooth over k and (3) is trivial [8, Proposition, p. 207]. We have a commutative diagram

$$\begin{tikzcd} N \arrow[r, "\delta"] \arrow[d] & \Omega_{A/k} \otimes_A B \arrow[d] \\ N_{\mathfrak{m}} \arrow[r, "\delta_{\mathfrak{m}}"] & \Omega_{A_{\mathfrak{m}}/k} \otimes_{A_{\mathfrak{m}}} B_{\mathfrak{m}} \end{tikzcd} \tag{4}$$

Since (4) is split, $\delta_{\mathfrak{m}}$ has a left inverse [8, Theorem 58, p. 187] and is, in particular, injective. By choice of $\mathfrak{m}$, $N \to N_{\mathfrak{m}}$ is injective, so we conclude that δ is nonzero. As previously noted, this implies that (1) is trivial. □

Corollary 1.3. *Let B be a simple G-algebra and let $f: A \to B$ be a surjective G-algebra homomorphism with nilpotent kernel. Then f has a G-algebra section.*

Proof. Proceed by induction on the exponent of nilpotence of $\operatorname{Ker} f$ as in the Hochschild–Whitehead theorem [6, Theorem 3.2, p. 286]. □

2. Equivariant inverse limits

The purpose of this section is to discuss projective limits for the category of $B.G$-modules. We use throughout the fact that the image of a G-linear homomorphism from a rational G-module to an abstract G-module is again a rational G-module, and that the image of a G- and B-linear homomorphism from a $B.G$-module to a simultaneous B and G module is again a $B.G$-module. In particular, if W is a rational G-submodule of a simultaneous B and G module Y, then BW (the image of $B \otimes W$ under multiplication) is a $B.G$-module. If V is an irreducible rational G-module and X an abstract G-module, then the sum X_V of all G-submodules of X isomorphic to V is a rational G-submodule of X isomorphic to $\mathrm{Hom}_G(V, X) \otimes V$ under the evaluation map $a \otimes v \to f(v)$. The direct sum $\bigoplus X_V$, as V ranges over the isomorphism classes of irreducible rational G-modules, is the maximal rational submodule of X, and it contains all other rational submodules.

It is possible to define inverse limits of $B.G$-modules for arbitrary index sets. We consider only limits indexed by the natural numbers $\mathbb{N}$.

Definition 2.1. Let $\{M_i, f_{ji}: M_j \to M_i \mid i, j \in \mathbb{N}\}$ be an inverse system of $B.G$-modules and homomorphisms. Then $^*\mathrm{proj\,lim}(M_i)$ denotes the largest rational submodule of $\mathrm{proj\,lim}(M_i)$, where the latter is the submodule of all coherent sequences in the abstract G-module $\prod M_i$

The following proposition is a direct consequence of the definition and our introductory remarks:

Proposition 2.2. *Let $\{M_i, f_{ji}\}$ be an inverse system of $B.G$-modules. Then*

(1) $^*\mathrm{proj\,lim}(M_i)$ *is a $B.G$-module*;

(2) *For each i the map $p_i: {}^*\mathrm{proj\,lim}(M_i) \to M_i$ induced from the ith projection of the product is a $B.G$-homomorphism*;

(3) *For any $B.G$-module N the maps p_i induce a bijection*

$$\mathrm{Hom}_{B.G}(N, {}^*\mathrm{proj\,lim}(M_i)) \to \mathrm{proj\,lim}\,\mathrm{Hom}_{B.G}(N, M_i);$$

(4) $({}^*\mathrm{proj\,lim}(M_i))^G = \mathrm{proj\,lim}(M_i^G)$.

Proof. For (1), we use that the maximal rational submodule of a simultaneous B and G module is a $B.G$-module. Part (2) is immediate, and for (3) we use that the image of a G-linear map from a $B.G$-module to an abstract G-module is rational to see that the alleged bijection is onto. For (4), we identify the left-hand side with $\mathrm{Hom}_{B.G}(B, {}^*\mathrm{proj\,lim}(M_i))$, use (3) to see that this is $\mathrm{proj\,lim}\,\mathrm{Hom}_{B.G}(B, M_i)$ and then identify $\mathrm{Hom}_{B.G}(B, M_i)$ with M_i^G. □

To study the properties of limits of $B.G$-modules, it is convenient to first treat the case $B = k$ (i.e., rational G-modules)

Proposition 2.3. *Let $\{X_i, f_{ji}\}$ be an inverse system of rational G-modules, and assume that the number of isomorphism classes of irreducible rational G-modules V with $f(X_i)_V \neq 0$ for some i is countable. Then*

(1) $\operatorname{proj}\lim(X_i) = \prod [\operatorname{proj}\lim((X_i)_V)]$;

(2) $\operatorname{proj}\lim((X_i)_V) = V \otimes \operatorname{proj}\lim(\operatorname{Hom}_G(V, X_i)) = {}^*\operatorname{proj}\lim((X_i)_V)$;

(3) ${}^*\operatorname{proj}\lim(X_i) = \bigoplus [V \otimes \operatorname{proj}\lim(\operatorname{Hom}_G(V, X_i))]$.

Proof. Let $\{V_j \mid j = 1, 2, \ldots\}$ be a list of the non-isomorphic irreducible G-modules occurring in all the X_i, and let $X_{ij} = (X_i)_{V_j}$ so $X_i = \bigoplus X_{ij}$. There is a natural map from the left side of (1) to the right, which is clearly injective. For any element $a = ((a_i)_1, (a_i)_2, \ldots)$ of the right side, define a sequence $b_i \in X_i$ by $b_i = (a_{i1}, a_{i2}, \ldots, a_{ii}, 0, 0 \ldots)$ where $a_{ij} \in X_{ij}$ is the ith entry of $(a_i)_j$. Then (b_i) is coherent and maps to a, so the natural map is an isomorphism. This proves (1). The first equality of (2) follows from the facts that $(X_i)_V = V \otimes \operatorname{Hom}_G(V, X_i)$ and that V, being finite dimensional over k, satisfies $\operatorname{proj}\lim(V \otimes \operatorname{Hom}_G(V, X_i)) = V \otimes \operatorname{proj}\lim \operatorname{Hom}_G(V, X_i)$. The second inequality in (2) comes from the fact that V tensored with a trivial module is rational. Part (3) then follows from (1) and (2). $\square$

The principal results of this section will apply to adic completions.

Theorem 2.4. *Let B be a Noetherian G-algebra and Q a $B.G$ ideal of B. Assume that the number of isomorphism classes of irreducible rational G-submodules of B is countable.*

(1) *Let $0 \to X' \to X \to X'' \to 0$ be an exact sequence of finitely generated $B.G$-modules. Then the following sequence is exact.*

$$
\begin{aligned}
0 \to {}^*\operatorname{proj}\lim(X'/Q^iX') &\to {}^*\operatorname{proj}\lim(X/Q^iX) \\
&\to {}^*\operatorname{proj}\lim(X''/Q^iX'') \to 0
\end{aligned}
$$

(2) *Let M be a finitely generated $B.G$-module. Then the following is an isomorphism:*

$$({}^*\operatorname{proj}\lim(B/Q^iB)) \otimes_B M \to {}^*\operatorname{proj}\lim(M/Q^iM)\,.$$

Proof. (1) A finitely generated $B.G$-module N is a homomorphic image of one of the form $B \otimes_k W$, where W is a finite-dimensional rational G-module. It follows that N and all its homomorphic images N/Q^iN only involve the same countable set of irreducible G-modules, so we can apply Proposition 2.3. By [1, Proposition

10.12, p. 108], the sequence of Q-adic completions of the sequence in (1) is exact. We denote these completions $\hat{}(\ )$ and their maximal rational submodules $^*\text{proj}\lim((\ )/Q^i(\ ))$ by $^*(\ )$. Then we have a commutative diagram with bottom row exact and vertical injections

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}^*X' & \longrightarrow & {}^*X & \longrightarrow & {}^*X'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \hat{}X' & \longrightarrow & \hat{}X & \longrightarrow & \hat{}X'' & \longrightarrow & 0 \end{array}$$

From Proposition 2.3, the top row is the direct sum and the bottom row the direct product of the sequences

$$\begin{aligned} 0 &\to \text{proj}\lim((X'/Q^iX')_V) \to \text{proj}\lim((X/Q^iX)_V) \\ &\to \text{proj}\lim(X''/Q^iX'') \to 0 \end{aligned}$$

The direct product of these being exact implies that their direct sum is exact and this proves (1).

(2) We use that M must be the cokernel of a $B.G$-module homomorphism $B \otimes_k P \to B \otimes_k Q$ where P and Q are finite-dimensional rational G-modules. By part (1), $^*(\ )$ preserves cokernels, as does tensoring with $^*B$, so we are led to the case $M = B \otimes_k P$ and the map $^*B \otimes_k P = {}^*B \otimes_B M \to {}^*M$. For any $B.G$-module N we have, using Proposition 2.2(3) and adjoint associativity, $\text{Hom}_{B.G}(N, {}^*B \otimes_k P) = \text{Hom}_{B.G}(N \otimes_k P^*, {}^*B) = \text{proj}\lim \text{Hom}_{B.G}(N \otimes_k P^*, B/Q^i) = \text{proj}\lim \text{Hom}_{B.G}(N, B/Q^i \otimes_k P) = \text{proj}\lim \text{Hom}_{B.G}(N, M/Q^iM) = \text{Hom}_{B.G}(N, {}^*M)$. It follows that the map is an isomorphism. □

We formalize the notation used in the proof of Theorem 2.4.

Definition 2.5. Let B be a G-algebra, Q a $B.G$-ideal of B and M a $B.G$-module. Then $^*M$ denotes $^*\text{proj}\lim(M/Q^iM)$ and is called the *rational Q-adic completion* of M. If M has at most a countable number of non-isomorphic irreducible rational G-submodules, we say that M is *countably typed.*

Theorem 2.4 does not claim that $^*B$ is Noetherian or that it is flat over B. The theorem does have some important consequences, however.

Corollary 2.6. *Let B be a countably typed Noetherian G-algebra, and let Q be a $B.G$-ideal of B. Then there are G-isomorphisms*

(1) $B/Q^i \to {}^*B/({}^*BQ)^i$,

(2) $\text{gr}_Q(B) \to \text{gr}_{{}^*BQ}({}^*B)$.

Proof. In the usual Q-adic completion $\hat{}(B/Q^i) = B/Q^i$ [1, p.109], so also $^*(B/Q^i) = B/Q^i$. From the exact sequence

$$0 \longrightarrow Q^i \longrightarrow B \longrightarrow B/Q^i \longrightarrow 0$$

and exactness of rational Q-adic completion Theorem 2.4(1) we have $^*(B/Q^i) = {}^*B/{}^*(Q^i)$, and by Theorem 2.4(2) $^*(Q^i) = {}^*BQ^i$. This gives the first isomorphism, and the second follows from the first in standard fashion. $\square$

As an example of rational Q-adic completions, we can consider the case where $B = S_k(W) = \bigoplus_{n\geq 0} S^n(W)$, the symmetric algebra on a finite-dimensional rational G-module W, and Q is the ideal $\bigoplus_{n\geq 1} S^n$. In this case the Q-adic completion is $\hat{}S_k(W) = \prod_{n\geq 0} S^n(W)$ and the rational Q-adic completion $^*S_k(W)$ is the largest G-rational submodule of $\hat{}S_k(W)$. When the G decomposition of all the $S^n(W)$ is known, $^*S(W)$ can be described using Proposition 2.3(3). It is also clear how to extend this example to symmetric algebras of $B.G$-modules when G acts trivially on B. We will deal with such algebras below.

3. Local structure of G-algebras

The purpose of this section is to describe the rational Q-adic completion at the ideal of a closed orbit, and its ring of invariants, primarily for the case of the coordinate ring of a non-singular affine variety with G-action.

Theorem 3.1. *Let B be a countably typed Noetherian B-algebra and Q a $B.G$-ideal of B, maximal among the $B.G$-ideals of B. Suppose also that Q is a local complete intersection ideal in B. Then there is a G-isomorphism $^*S_{B/Q}(Q/Q^2) \to {}^*B$.*

Proof. Let $B_0 = B/Q$. Then B_0 is $B_0.G$-simple, so every $B_0.G$-module is B_0- (and $B_0.G$-) projective [4, Proposition 3.5, p. 794]. Let $N = Q/Q^2$. N is a finitely generated B_0-module, hence projective. Thus to say that Q is a local complete intersection ideal means only to assume that $\beta : S = S_{B_0}(N) \to \mathrm{gr}_Q(B)$ is an isomorphism. To define a map from S to $^*B$, it will suffice by Proposition 2.2(3) to define consistent maps from S to B/Q^i for each i. Constructing such a G-algebra map is equivalent to finding a G-algebra map $s_i : B_0 \to B/Q^i$ and a $B_0.G$-module map $f_i : N \to B/Q^i$. Take $s_1 = \mathrm{id}$ and $f_1 = 0$. For the case $i = 2$, we have the sequence $0 \to N \to B/Q^2 \to B_0 \to 0$. We can take f_2 to be the inclusion and s_2 the section guaranteed by Theorem 1.2. Suppose f_i, s_i to be defined. Let $h : B/Q^{i+1} \to B/Q^i$ be the quotient map. Then $h^{-1}(s_i(B_0))$ is a square zero extension of B_0, hence trivial by Theorem 1.2. If t is a G-section, take s_{i+1} to be t followed by inclusion to B/Q^{i+1}. Then $hs_{i+1} = s_i$, so s_{i+1} is consistent. We can then regard h as a surjection of $B_0.G$-modules, and as such it is split. Let r be a

$B_0.G$-inverse to h and let $f_{i+1} = rf_i$. Then f_{i+1} consistently extends f_i. Thus for each i we obtain a pair (s_i, f_i). These pairs define a coherent sequence of G-algebra homomorphisms $\alpha_i : S \to B/Q^i$. Let P denote the ideal of S generated by N. By construction, α_2 induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N = P/P^2 & \longrightarrow & S/P^2 & \longrightarrow & S/P = B_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N & \longrightarrow & B/Q^2 & \longrightarrow & B/Q = B_0 & \longrightarrow & 0 \end{array}$$

Since α_i extends α_2, we see that $\alpha_i(P) \subseteq Q/Q^2$ and that the map $P/P^2 \to Q/Q^2$ induced from α_i is the same as that induced from α_2, i.e. the identity. It follows that

$$\beta_i : \mathrm{gr}_{P/P^i}(S/P^i) \to \mathrm{gr}_{Q/Q^i}(B/Q^i)$$

induced from α_i is just the projection on the first $i+1$ terms of β, and hence is an isomorphism. Since S/P^i is (P/P^i)-adically complete and B/Q^i is (Q/Q^i)-adically complete, we conclude that α_i induces an isomorphism $\gamma_i : S/P^i \to B/Q^i$ of G-algebras. Then $^*\mathrm{proj\,lim}(\gamma_i)$ provides the isomorphism of the theorem. □

In case B is the coordinate ring of an affine variety with G-action, Theorem 3.1 takes the following geometric form:

Corollary 3.2 (Local linearization). *Let X be an affine variety over K with G-action. Let Gx be a closed orbit in X and let H be the* (*reductive*) *stabilizer of x. Assume that Gx is a local complete intersection in X and let Q be its ideal in $k[X]$. Then there is a finite-dimensional H-module V such that the rational Q-adic completion $^*k[X]$ is G-isomorphic to $^*S_{k[G/H]}(V|_H^G)$.*

Proof. $k[X]$ is countable-dimensional over k, hence countably typed, and a Noetherian G-algebra. The orbit $Gx = G/H$ has coordinate ring $k[X] = k[G/H]$. The $k[G/H].G$-module Q/Q^2 must be of the form $V|_H^G$ by [9, Theorem 3.1, p. 42]. Then Corollary 3.2 follows from Theorem 3.1. □

In applying Corollary 3.2, it is worth noting that for X non-singular the non-singularity of a closed orbit implies that it is a local complete intersection [8, Theorem 36, p. 121 and Theorem 33, p. 114]. Further, we note that the algebra $^*S(V|_H^G)$ is defined purely in terms of G (no X). Thus possible rational adic completions of non-singular varieties at ideals of closed orbits are all obtained from rational completions of algebras of the form $^*S_{k[G/H]}(V|_H^G)$. This is the source of the name of Corollary 3.2. Finally, we note that the module V in Corollary 3.2 (and in Corollaries 3.3 and 3.4 below) obtained from Q/Q^2 by [9, Theorem 3.1, p. 42] can be geometrically described as the fibre over x of the conormal bundle of Gx in X.

Corollary 3.3 (Local structure of invariants). *Let X be an affine variety over k with G-action, let Gx be a closed orbit in X, and let H be the (reductive) stabilizer of x in G. Assume that Gx is a local complete intersection in X and let Q be its ideal in $k[X]$. Then there is a finite-dimensional H-module V such that the Q^G-adic completion of $k[X]^G$ is isomorphic to ${}^*S_k(V)^G = {}^\wedge S_k(V)^G$.*

Proof. Let $B = k[X]$, $C = k[X]^G$, and $\mathfrak{m} = Q^G$. The $\mathfrak{m}$-adic completion of C is $\operatorname{proj\,lim}(C/\mathfrak{m}^i)$. Let $Q_i = Q^i \cap C$, so $(B/Q^i)^G = C/Q_i$. Then $Q_1 = \mathfrak{m}$ is maximal and the Q_i are cofinal in the $\mathfrak{m}^i$. It follows that the $\mathfrak{m}$-adic completion of C is also $\operatorname{proj\,lim}(C/Q_i) = [{}^*\operatorname{proj\,lim}(B/Q^i)]^G$, the last equality by Proposition 2.2(4). By Corollary 3.2, the right-hand side is $\operatorname{Hom}_{B,G}(B, {}^*S_{B/Q}(V|_H^G))$. By Proposition 2.2(3), and the fact that $S_{B/Q}(V|_H^G) = S_k(V)|_H^G$, we can by the categorical isomorphism [7] rewrite the Hom as $\operatorname{Hom}_H(k, {}^*S_k(V)) = \operatorname{Hom}_H(k, {}^\wedge S_k(V))$, and the corollary follows. $\square$

For non-singular X, Corollary 3.3 applies as noted above to all closed orbits. Since every maximal ideal of $k[X]^G$ comes from an ideal of a closed orbit, we obtain

Corollary 3.4. *Let X be a non-singular affine variety over k with G-action, let C be the ring of invariants $k[X]^G$, and let $\mathfrak{m}$ be a maximal ideal of C. Then there is a reductive subgroup H of G and a finite-dimensional H-module V such that the $\mathfrak{m}$-adic completion of C is isomorphic to ${}^\wedge S_k(V)^H = \prod_{n\geq 0} S^n(V)^H$.* $\square$

4. Simple algebras and homogeneous spaces

In this section, which is independent of the rest of the paper, we show that a simple G-algebra is a form of an affine homogeneous G-space. In particular, this shows that a G-simple algebra is necessarily of finite type over its field of invariants. These results are based on discussions with Singer; an important special case can be found in his paper [10, Lemma 4]. N.B., in this section G is not required to be reductive.

This section uses base change. To explain this conveniently, we need to regard G-algebras from a comodule point of view. We recall that a G-algebra structure on R is equivalent to a k-algebra morphism

$$\gamma_R = \gamma : R \to R \otimes_k k[G]$$

(where $\gamma(r) = \sum r_i \otimes f_i$ if $g(r) = \sum f_i(g) r_i$ for all g in G), making R into a

$k[G]$-comodule. We note that γ is also a G-module homomorphism, where in the tensor product G acts trivially on R. If K is an (algebraically closed) extension field of k, then $K\otimes_k \gamma$ makes $K\otimes_k R$ into a $K\otimes_k k[G]$-comodule, or a G_K algebra, where G_K is G base-changed to K. We can, in particular, choose a field K such that R has a K-point, i.e. such that there is a k-algebra homomorphism $\eta: R\to K$. We retain the meaning of γ and η throughout this section.

Lemma 4.1. *A simple G-algebra R is an integral domain.*

Proof. The composite $(\eta\otimes 1)\gamma$ is a G-algebra homomorphism and hence an embedding of R in the integral domain $K\otimes k[G]=K[G_K]$. □

Since a simple G-algebra R is a domain, we can discuss its quotient field E. The G-action on R extends to a (non-rational) action on E.

Lemma 4.2. *Let R be a G-simple algebra with quotient field E. Then R is the maximal rational G-submodule of E.*

Proof. Let $f\in E$ and suppose the k-span V of $\{g(f)\mid g\in G\}$ is finite-dimensional. Let $I=\{h\in R\mid hV\subset R\}$. Then I is a G-stable ideal of R, and is non-zero since it contains a common denominator for a basis of V. Thus $I=R$ and $f=1f$ is in R. □

Lemma 4.3. *Let R be a G-simple algebra. Then $L=R^G$ is a field and $\gamma: R\to R\otimes_k k[G]$ is L-linear.*

Proof. If a is a non-zero element of L, then Ra is non-zero and G-stable, hence equals R, so a is a unit. The inverse of an invariant unit is invariant, so L is a field. If r, s, and g are in R, L and G, and $\gamma(r)=\Sigma\, r_i\otimes f_i$, then $g(sr)=sg(r)=\Sigma\, f_i(g)sr_i$, so $\gamma(sr)=\Sigma\, sr_i\otimes f_i=(s\otimes 1)\gamma(r)$. □

From Lemma 4.3 we see that R is an L-algebra on which the L-Hopf algebra $L\otimes_k k[G]$ co-acts via $\gamma: R\to R\otimes_L (L\otimes k[G])$, and R is $L\otimes k[G]$-simple in the obvious sense.

Scalar extension of a G-algebra need not be simple; consider the case that G is trivial and R a field. However, we do have the basic result that the G-stable ideals in scalar extensions contain invariants:

Lemma 4.4. *Let R be a G-simple algebra, L be R^G, and A a commutative L-algebra. Let G act on $A\otimes_L R$ via $1\otimes G$. Then a non-zero G-stable ideal I of $A\otimes_L R$ contains a non-zero element of A.*

Proof. Suppose n is chosen minimal such that I contains a non-zero element $y = b_1 \otimes r_1 + \cdots + b_n \otimes r_n$ with the b_i linearly independent over L. Let $J = \{r \in R \mid \text{there are } s_2, \ldots, s_n \in R \text{ with } b_1 \otimes r_1 + \sum b_i \otimes s_i \text{ in } I\}$. Then J is a G-stable ideal of R containing r_1, so $J = R$. Thus I contains an element $z = b_1 \otimes 1 + \sum b_i \otimes s_i$. For $g \in G$, $z - g(z) = \sum_2^n b_i \otimes (s_i - g(s_i))$ is shorter than y, and in I, so is zero and $z = g(z)$. Of course the linear independence of the b_i shows that z is non-zero, and $z \in (A \otimes_L R)^G = A$. $\square$

We can now prove the main theorem.

Theorem 4.5. *Let R be a G-simple algebra. Then there is an algebraically closed field extension K of $L = R^G$ such that $K \otimes_L R$ is G_K-isomorphic to the coordinate ring of an affine homogeneous space for G_K.*

Proof. In our selection of the point η, we choose an algebraically closed extension K of L. By Lemma 4.4, $K \otimes_L R$ is G- (and hence G_K-) simple, and it has a K-point $1 \otimes \eta$. Replacing k by K, G by G_K, R by $K \otimes_L R$, η by $1 \otimes \eta$, and γ by $K \otimes_L \gamma$, we may assume that R has a k-point η and that $R^G = k$. Then $(\eta \otimes 1)\gamma$ is a G-embedding of R in $k[G]$, and we view $R \subset k[G]$ as a subalgebra. (G acts on $k[G]$ via $g(f)(x) = f(xg)$.) We can write R as an ascending union of G-algebras R_i, each finitely generated over k. Let $H_i = \{x \in G \mid f(x) = f(e) \text{ for all } f \in R_i\}$ and let $\bigcup H_i = H$. The H_i form a descending chain of closed subgroups of G, so there is an i_0 such that $H_i = H$ for all $i \geq i_0$. Since R is a G-algebra, if f, h, and g are in R, H, and G, then $g(f)(h) = g(f)(e)$ or $f(hg) = f(g)$. That is f is constant on the right cosets Hg, so $R \subset k[H\backslash G]$. Let X_i be the variety with $k[X_i] = R_i$. The inclusion $R_i \subset k[G]$ defines a dominant equivariant morphism $\phi : G \to X_i$ whose image is open. If $x_i = \phi(e)$, then the stabilizer of x_i is H_i and the image of ϕ is isomorphic to $H_i\backslash G$ via $H_i g \to x_i g$. Since this isomorphism is birational, the quotient field E_i of R_i is the subfield $k(H_i\backslash G)$ of the field $k(G)$. Taking $i \geq i_0$ we see that the quotient field E of R, which lies between E_i and $k(H\backslash G)$, must equal $k(H\backslash G)$. Since $k[H\backslash G]$ is a rational submodule of $k(H\backslash G)$ we have by Lemma 4.2 that $k[H\backslash G] \subset R$, and so $R = k[H\backslash G]$. Let S denote the $R.G$-algebra $k[G]$. Since k has characteristic zero, the E-algebra $(S \otimes_R E) \otimes_E (S \otimes_R E)$ is reduced, so the nilradical N of $S \otimes_R S$ is R-torsion. But N is a union of projective R-modules, hence torsion free, by [4, Proposition 2.2, p. 790], hence $N = 0$. The same result also implies that S is faithfully flat over R. These facts about S and $S \otimes_R S$ allow us to apply the proof of [3, Theorem 4.3, p. 9] to conclude that $H\backslash G$ is affine. The theorem follows. $\square$

Since finite generation is refected by flat base change, we further obtain

Corollary 4.6. *Let R be a G-simple algebra. Then R is finitely generated over the field R^G. In particular, R is Noetherian.* $\square$

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